ST227 - Exercise 2

Question 1

The lifetime of a mechanical system is modelled by the following mortality intensity:

$$\mu(t) = \lambda, \quad t \geq 0, \quad \lambda = 10^{-1}\,.$$

a. Define in R the survival probability function $(t, x) \to {}_tp_x$ and calculate the probability of surviving the next 5 years for a 15-year-old system.

   Recall that

$$ {}_tp_x = \exp\left\{-\int_x^{t+x} \mu(t)dt\right\}\,.$$

```
lambda = 1e-1
mu <- function(t){vapply(t, function(t){lambda}, 0.0)}

tpx <- function(t, x){
    exp(-integrate(mu, lower = x, upper = x + t)$value)
}

tpx(t = 5, x = 15)
```

```
## [1] 0.6065307
```

Here, we needed to vectorize the function $\mu(t)$ using `vapply` for the numerical integration with `integrate`. Alternatively, you can use the `quad` function from the `pracma` package, in which case you do not need to vectorize $\mu(t)$.

General wear and tear in mechanical systems mean the memoryless property of constant mortality is a questionable one. We overcome that by introducing a time-dependent term:

$$\tilde{\mu}(t) = \lambda + \gamma \log(\log(e + t)), \quad t \geq 0, \quad \lambda = 10^{-1}, \gamma = 1.5\,.$$

Consider a 15-year-old machine, of which the remaining lifetime T_{15} follows the mortality function $\tilde{\mu}(t)$.

b. Define in R the density function for T_{15}. This definition may involve a numerical integral.

   The construction of the ${}_tp_x$ function is similar as in 1.a. The density is given by

$$T_{15}(t) = {}_tp_x(t, 15)\mu(t + 15)\,.$$

```
lambda <- 0.1
gamma <- 1.5

mu <- function(t){
    vapply(t, function(t){lambda + gamma * log(log(exp(1) + t))}, 0.0)
}

tpx <- function(t,x){
    exp(-integrate(mu, lower = x, upper = x + t)$value)
```

```r
}

density <- function(t){tpx(t = t, x = 15) * mu(t + 15)}
```

c. Calculate the expected remaining lifetime for this individual.

   We compute
$$\int_0^\infty tT_{15}(t)dt\,.$$

```r
integrand <- function(t){vapply(t, function(t){t * density(t)}, 0.0)}
integrate(integrand, lower = 0, upper = Inf)
```

```
## 0.5881207 with absolute error < 1.4e-05
```

d. Define in R the cumulative distribution function of T_{15}. This definition may involve a numerical integral. Discuss how you would find the 95-th percentile of T_{15}.

   The CDF ${}_tq_x$ is one minus the hazard function ${}_tp_x$. We find the 95-th percentile of it using the `uniroot` function.

```r
tqx <- function(t, x){
    1-tpx(t = t, x = x)
}

diff <- function(t){tqx(t = t, x = 15) - 0.95}

root <- uniroot(diff, c(0, 100))$root

root
```

```
## [1] 1.753325
```

```r
tqx(root, 15)
```

```
## [1] 0.9500002
```

Question 2

Consider the constant morstality function
$$\mu(t) = 0.05\,.$$

a. Calculate the probability that individuals aged 20, 40 and 80 will survive the next 20 years.

```r
mu <- function(t){
    vapply(t, function(t){0.05}, 0.0)
}
tpx <- function(t, x){
    exp(-integrate(mu, lower = x,upper = t + x)$value)
}
tpx(t = 20, x = 20)
```

```
## [1] 0.3678794
```

```r
tpx(t = 20, x = 40)
```

```
## [1] 0.3678794
```

```r
tpx(t = 20, x = 80)
```

```
## [1] 0.3678794
```

A class of commonly considered mortality is called Gompertz-Makeham's (or simply Makeham's) mortality, which has the form:

$$A + Bc^x\,.$$

b. Using Makeham's mortality with $A = 5 \times 10^{-4}$, $B = 7.5858 \times 10^{-5}$ and $c = 1.09144$, calculate the probability that individuals aged $20, 40$ and 80 will survive the next 20 years.

```r
A <- 5e-04
B <- 7.5858e-05
c <- 1.09144
mu <- function(t){
    vapply(t, function(t){A+B*c^t}, 0.0)
}
tpx <- function(t, x){
    exp(-integrate(mu, lower = x, upper = t + x)$value)
}
tpx(t = 20, x = 20)
```

```
## [1] 0.9668433
```

```r
tpx(t = 20, x = 40)
```

```
## [1] 0.8637371
```

```r
tpx(t = 20, x = 80)
```

```
## [1] 0.01078663
```

c. Denote by K_x the remaining curtate lifetime of an individual aged x . Then

$$K_x = \lfloor T_x \rfloor\,,$$

where the mapping $t \to \lfloor t \rfloor$ is the floor function. Calculate the curtate lifetime of a 20-year old individual with Makeham's mortality function.

The expected curtate lifetime is

$$\mathrm{E}(K_{20}) = \int_0^\infty \lfloor t \rfloor {}_tp_{20}\mu(20+t)dt\,.$$

```r
integrand <- function(t){
    vapply(t, function(t){floor(t) * mu(20 + t) * tpx(t, 20)}, 0.0)
}

integrate(integrand, lower = 0, upper = 100)
```

```
## 53.03114 with absolute error < 0.0025
```

d. It can be shown that

$$\mathrm{E}(K_x) = \sum_{i=1}^{\infty} {}_np_x\,.$$

Utilise this formula to calculate the expected curtate lifetime. Approximate the infinite sum by the first 100 terms.

```r
probs <- vapply(1:100,function(n){tpx(n, 20)}, 0.0)
sum(probs)
```

```
## [1] 53.03153
```

Question 3

Consider the following mortality intensity function

$$\mu(t) = \frac{\alpha\gamma t^{\gamma-1}}{1+\alpha t^{\gamma}}, \quad \alpha = 3.757 \times 10^{-2.2}, \quad \gamma = 1.4243\,.$$

a. Define in R the survival probability function $(t, x) \to {}_tp_x$ and calculate the probability of surviving the next 3 years for a 20-year-old individual.

```r
alpha <- 3.757 * 10^(-2.2)
gamma <- 1.4243

mu <- function(t){
    vapply(t, function(t){(alpha * gamma * t^(gamma - 1)) / (1 + alpha * t^gamma)}, 0.0)
}
tpx <- function(t, x){
    exp(-integrate(mu, lower = x,upper = t + x)$value)
}
tpx(t = 3, x = 20)
```

```
## [1] 0.8784416
```

b. Define in R the density function for T_{20}.

```r
density <- function(t){tpx(t = t, x = 20) * mu(t + 20)}
```

c. Calculate the expected remaining lifetime for a 20-year-old individual.

```r
integrand <- function(t){vapply(t, function(t){t * density(t)}, 0.0)}
integrate(integrand, lower = 0, upper = Inf)
```

```
## 67.27186 with absolute error < 0.0018
```

d. Explain how you would find the median of this distribution.

```r
tqx <- function(t, x){
    1 - tpx(t = t, x = x)
}

diff <- function(t){tqx(t = t, x = 20) - 0.5}

root <- uniroot(diff, c(0, 100))$root

root
```

```
## [1] 19.03081
```

```r
tqx(root, 20)
```

```
## [1] 0.5
```

Question 4

The time (in days) to failure, denoted T_x, of a mechanical system aged x can can be mod-lled by the following mortality (also known as the hazard function in non-life contexts)

$$\mu(t) = \lambda, \quad t \geq 0, \lambda = 0.05\,,$$

a. Define in R the survival probability function $(t, x) \to {}_tp_x$ and calculate the probability that a new system will operate without failures for 10 days.

```
lambda = 0.05
mu <- function(t){vapply(t, function(t){lambda}, 0.0)}

tpx <- function(t, x){
    exp(-integrate(mu, lower = x, upper = x + t)$value)
}

tpx(t = 0, x = 10)
```

```
## [1] 1
```

b. Define in R the density function for T_0.

```
density <- function(t){tpx(t = t, x = 0) * mu(t)}
```

c. Calculate the expected number of days between until failure of a new system and that of a system that has been continuously running for 20 days. Comment on your finding.

```
integrand <- function(t){vapply(t, function(t){t * density(t)}, 0.0)}
e_0 <- integrate(integrand, lower = 0, upper = Inf)$value

density20 <- function(t){tpx(t = t, x = 20) * mu(t + 20)}
integrand <- function(t){vapply(t, function(t){t * density20(t)}, 0.0)}
e_20 <- integrate(integrand, lower = 0, upper = Inf)$value

cbind(e_0, e_20)
```

```
##      e_0 e_20
## [1,]  20   20
```

They are the same as the hazard function is constant over time. The expected time to failure is the reciprocal of λ in this case.

d. Consider a modified version of the hazard function

$$\mu(t) = \lambda + 0.2\log(t+1)\,.$$

Compute $\mathrm{E}(T_0)$ and $\mathrm{E}(T_{20})$ again. Comment on their values and interpret the impact of the extra term in the modified hazard function.

```
lambda = 0.05
mu <- function(t){vapply(t, function(t){lambda + 0.2 * log(t + 1)}, 0.0)}

tpx <- function(t, x){
    exp(-integrate(mu, lower = x, upper = x + t)$value)
}

density <- function(t){tpx(t = t, x = 0) * mu(t)}
integrand <- function(t){vapply(t, function(t){t * density(t)}, 0.0)}
```

```r
e_0 <- integrate(integrand, lower = 0, upper = Inf)$value

density20 <- function(t){tpx(t = t, x = 20) * mu(t + 20)}
integrand <- function(t){vapply(t, function(t){t * density20(t)}, 0.0)}
e_20 <- integrate(integrand, lower = 0, upper = Inf)$value

cbind(e_0, e_20)
```

```
##            e_0     e_20
## [1,] 3.622364 1.488132
```

The time dependent hazard function results in $\mathrm{E}(T_0)$ being larger than $\mathrm{E}(T_2 0)$, which seems a more reasonable modelling assumption for most cases.